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Theory of thermal resistance between solids with randomly sized and located contacts

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Abstract

Linear superposition method is used to determine an analytical solution of the thermal constriction resistance adapted to random contacts. The contact area is constituted of numerous disks which have different radii and are randomly distributed over a square or circular area. The developed solution is easy to use and allows to consider numerous contacts at a reasonable computing time. The disks can be distributed in entire contact surface or located in a specific region. The results are in an excellent agreement with available data in the literature for identical and regular contacts. The model is used to study the thermal constriction resistance evolution as a function of contact disorders, number and sizes of disks and relative contact size area. The results are compared to the model of regular contacts. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Thermal contact resistance; Thermal constriction; Random contacts; Interface; Heat conduction; Heat sources; Analytical calculation

1. Introduction

Thermal constriction resistance has been widely studied over these last decades. Numerous models [1–4] have been developed to calculate its value as a function of geometrical mechanical and thermal characteristics of materials. These models are generally based on idealized contacts (identical asperities, regularly distributed over the contact plane). Actually, the size of real contact areas is different and the contacts are randomly distributed. An experimental study [1], using electrical analogy has been performed to examine the effect of the eccentricity of unique contact on the evolution of the constriction resistance. An analytical related solution [5] has been developed. A good agreement was obtained between these results and experimental data. It shows that the thermal constriction resistance increases with the increasing of the eccentricity and this effect is more important when the relative contact size ε value is high. This analytical solution has been generalized to multiple contacts with different sizes, randomly distributed over the surface of a laterally insulated semi-infinite square prism. In this model each contact is submitted to an uniform heat flux of which the value is determined by the writing of the equality of the average temperatures of all contacts. The analytical solution of this problem [6] has been developed using the finite cosine Fourier transform and its inverse. The solution includes a double series which slowly converges and needs an extensive computing time. Therefore it was difficult to consider an important number of contacts. The results show that the thermal constriction resistance for random contacts is systematically greater than that for identical and regular contacts. A recent study [7] confirms this trend for identical disks randomly distributed over a square contact region.

An analytical solution that accounts for the contact disorder is developed in this paper. The contacts are modeled by multiple disks with different radii. The solution is established using the linear superposition

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Nomenclature

A_a	apparent contact area	L	length of square contact region
A_r	real contact area	N	number of contacts
a	radius of disk contact	q	heat flux
b	radius of circular contact region	R_{cs}	thermal constriction resistance
b_{ij}	distance between centers of contacts i and j	RR	ratio of maximum to minimum disk radii, $= \mathrm{Max}(a)/\mathrm{Min}(a)$
E	relative difference between random and regular contacts (Eq. 19)	r, θ, z	space polar coordinates
e	distance between centers of cylinder and contact zone	T	temperature
		x, y, z	space cartesian coordinates
k	thermal conductivity	ε	relative contact size, $=\sqrt{A_r/A_a}$

method. This solution is easy to use and allows to treat numerous contacts (a few hundreds) at a reasonable computational time compared to that of Ref. [6]. The model can be adapted to Hertzian contacts (Fig. 1a) or large contact surface (Fig. 1b). The results for random and regular contacts are compared considering the same number of contacts and the same real contact area. The thermal constriction resistance evolution is studied as a function of the relative contact size, the contact disorder, the number of contacts and the radii dispersion (ratio of maximum to minimum spots radii).

2. Analytical model

In order to establish the model, we consider a semi-infinite medium (Fig. 2) with a thermal conductivity k and a zero reference temperature. The surface ($z = 0$) is submitted to N heat sources (j) of circular shape (radius a_j) generating an uniform flux q_j in order to simulate the contact with an other solid. The heat sources are randomly distributed over a square (Fig. 2a) or circular (Fig. 2b) region. The rest of this surface is insulated. The heat transfer due to all sources is three-dimensional. Since the heat conduction equation is linear, we first

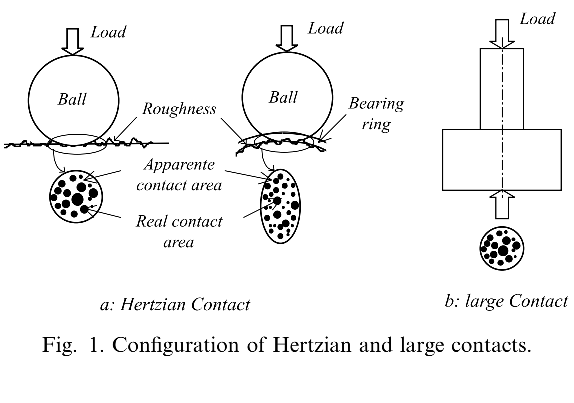

Fig. 1. Configuration of Hertzian and large contacts.

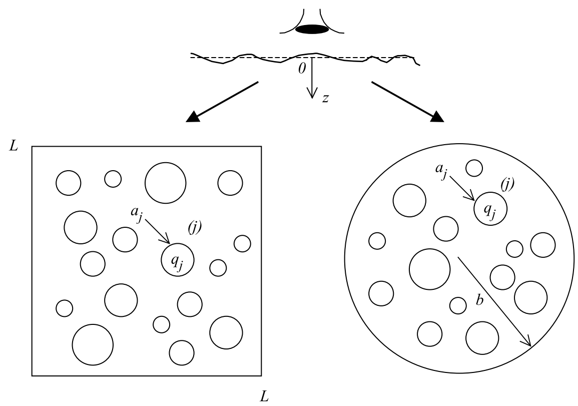

Fig. 2. Geometry of studied model.

determine the response for a unique source (two-dimensional problem, $T(r, z)$). We then use the superposition method to account for other contacts.

The heat transfer governing equations due to a unique heat source (j) are the following:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_j}{\partial r}\right) + \frac{\partial^2 T_j}{\partial z^2} = 0 \tag{1}$$

$$\left.\frac{\partial T_j}{\partial r}\right|_{r=0} = 0 \tag{2}$$

$$-k\left.\frac{\partial T_j}{\partial z}\right|_{z=0} = \begin{cases} q_j & \text{(at contact)} \\ 0 & \text{(elsewhere)} \end{cases} \tag{3}$$

$$T_j\big|_{r\to\infty, z\to\infty} = 0 \tag{4}$$

The solution of this problem is classic and is obtained by applying the infinite Hankel transform and its inverse. It allows to write the surface temperature as

$$T_j(r, 0) = \frac{q_j a_j}{k}\int_0^\infty \frac{1}{\beta} J_1(\beta a_j) J_0(\beta r)\,\mathrm{d}\beta \tag{5}$$

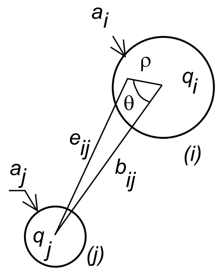

Fig. 3. Principle of superposition method.

At the radius $r > a_j$, the integral (5) can be written as [8, p. 715]:

$$T_j(r,0) = \frac{q_j a_j}{2k} F\left(\frac{1}{2},\frac{1}{2};2;\left(\frac{a_j}{r}\right)^2\right) \tag{6}$$

where $F(\alpha,\beta;\gamma;Z)$ is an hypergeometric function defined by

$$F(\alpha,\beta;\gamma;Z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{Z^n}{n!} \tag{7}$$

and Γ is the gamma function. Replacing (7) into (6) and expanding the series, $T_j(r,0)$ can be expressed as follows:

$$T_j(r,0) = \frac{q_j a_j}{k}\left[\frac{1}{2}\left(\frac{a_j}{r}\right) + \frac{1}{16}\left(\frac{a_j}{r}\right)^3 + \frac{3}{128}\left(\frac{a_j}{r}\right)^5 + \frac{25}{2048}\left(\frac{a_j}{r}\right)^7 + \cdots\right] \tag{8}$$

At the distance $r = e_{ij}$ (Fig. 3), where

$$e_{ij} = \sqrt{\rho^2 + b_{ij}^2 - 2\rho b_{ij}\cos\theta} \tag{9}$$

the ratio (a_j/e_{ij}) can be expressed using Legendre polynomials [8, p. 1048] as follows:

$$\frac{a_j}{e_{ij}} = \frac{a_j}{b_{ij}\sqrt{(\rho/b_{ij})^2 + 1 - 2(\rho/b_{ij})\cos\theta}} = \frac{a_j}{b_{ij}} \sum_{k=0}^{\infty}\left(\frac{\rho}{b_{ij}}\right)^k P_k(\cos\theta) \tag{10}$$

Replacing (a_j/e_{ij}) by its expression in Eq. (8) and superposing temperatures due to contacts (j) to that of the contact (i), the average temperature of contact (i) can be written as

$$\overline{T}_{\mathrm{c}}^{(i)} = \frac{8q_i a_i}{3k\pi} + \sum_{\substack{j=1\\(j\neq i)}}^{N} \frac{q_j a_j}{k}\left(\frac{a_j}{b_{ij}}\right)\left\{\frac{1}{2} + \frac{1}{16}\left(\frac{a_j^2 + a_i^2}{b_{ij}^2}\right) + \frac{9}{128}\left(\frac{a_j^4 + 3a_i^2 a_j^2 + a_i^4}{b_{ij}^4}\right)\right\} + O\left(\frac{a_{i,\mathrm{or},j}}{b_{ij}}\right)^7 \tag{11}$$

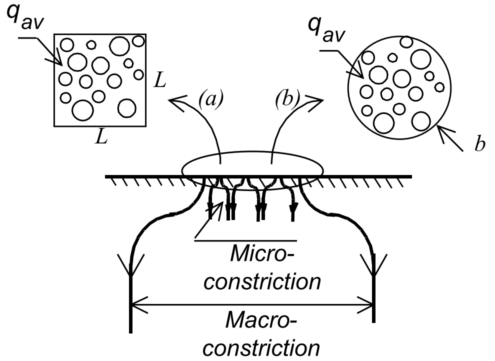

Fig. 4. Description of micro and macro-constriction phenomena.

In Eq. (11) the expanding series is stopped at $(a_{i,\mathrm{or},j}/b_{ij})^5$ which is widely sufficient because in practice $a_{i,\mathrm{or},j} \ll b_{ij}$. The first term of Eq. (11) corresponds to the effect of the contact (i) only and the second one presents the interaction between all the contacts (j) with the contact (i).

The individual thermal constriction resistance, $R_{\mathrm{cs}}^{(i)}$, due to the contact (i) is defined by

$$R_{\mathrm{cs}}^{(i)} = \frac{\Delta\overline{T}_{\mathrm{c}}^{(i)}}{q_i\pi a_i^2} = \frac{\overline{T}_{\mathrm{c}}^{(i)} - T_{\mathrm{c},i}}{q_i\pi a_i^2} \tag{12}$$

where $\overline{T}_{\mathrm{c}}^{(i)}$ is the average temperature of contact (i) and $T_{\mathrm{c},i}$ the temperature at the center of contact (i) due to a macro-constriction phenomenon which occurs around the apparent contact area A_{a} (Fig. 4). This temperature can be estimated considering the same contact region subjected to the average heat flux $q_{\mathrm{av}} = \sum_{j=1}^{N}(q_j A_j)/A_{\mathrm{a}}$. In a practical point of view, taking into account of the contact disorder, the macro-constriction is not veritably due to an uniform heat flux. We have analyzed this assumption in comparing the results of the present model to that [6] adapted to a contact of a finite surface. The difference between these two models is tiny (<10%). The expression of $T_{\mathrm{c},i}$ due to a square heat source is given in Ref. [9]:

$$\begin{aligned} T_{\mathrm{c},i} = \frac{q_{\mathrm{av}}}{2\pi k}\Bigg\{ &(x_i+L)\ln\left[\frac{(y_i+L)+\sqrt{(x_i+L)^2+(y_i+L)^2}}{(y_i-L)+\sqrt{(x_i+L)^2+(y_i-L)^2}}\right] \\ &-(x_i-L)\ln\left[\frac{(y_i+L)+\sqrt{(x_i-L)^2+(y_i+L)^2}}{(y_i-L)+\sqrt{(x_i-L)^2+(y_i-L)^2}}\right] \\ &+(y_i+L)\ln\left[\frac{(x_i+L)+\sqrt{(x_i+L)^2+(y_i+L)^2}}{(x_i-L)+\sqrt{(x_i-L)^2+(y_i+L)^2}}\right] \\ &-(y_i-L)\ln\left[\frac{(x_i+L)+\sqrt{(x_i+L)^2+(y_i-L)^2}}{(x_i-L)+\sqrt{(x_i-L)^2+(y_i-L)^2}}\right]\Bigg\} \end{aligned} \tag{13}$$

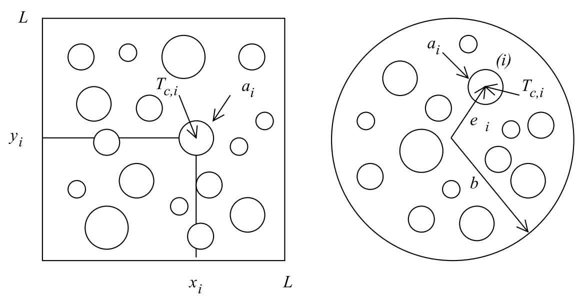

Fig. 5. Parametrizing of the macro-constriction region.

where x_i and y_i are the cartesian space coordinates (Fig. 5) of the center position of contact (i).

In the case of circular apparent contact area which b radius, the temperature $T_{c,i}$ can be calculated using Eq. (5), replacing a_i by b and r by e_i, where e_i is the polar position of the center of contact (i) from the center of contact region. We have

$$T_{c,i} = T(r = e_i) = \frac{q_{av} b}{k} F\left(\frac{1}{2}, -\frac{1}{2}; 1; \left(\frac{e_i}{b}\right)^2\right) \tag{14}$$

To calculate the thermal constriction resistance R_{cs} due to all contacts, we put $(\overline{T}_c^{(i)} - T_{c,i}) = \Delta T_c = \mathrm{Cste}\ (\forall i)$ in the Eq. (12) and we write

$$R_{cs} = \left[\sum_{i=1}^{N} \frac{1}{R_{cs}^{(i)}}\right]^{-1} \tag{15}$$

Eqs. (11), (13) and (14) can be written as follows:

$$\overline{T}_c^{(i)} = G_{ii} q_i + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} G_{ij} q_j \tag{16}$$

$$T_{c,i} = H_i q_{av} = H_i \sum_{j=1}^{N} \frac{\pi a_j^2}{A_a} q_j \tag{17}$$

The terms G_{ii} and G_{ij} are given in Eq. (11). The term H_i is given in Eq. (13) for a square area (where $A_a = L^2$) and Eq. (14) for a circular area (where $A_a = \pi b^2$).

By writing the equality $(\overline{T}_c^{(i)} - T_{c,i}) = \Delta T_c$, we obtain a linear matrix system of order N whose unknowns are $(q_j/\Delta T_c)$. This system is written in the form

$$\left(G_{ii} - H_i \frac{\pi a_i^2}{A_a}\right) \frac{q_i}{\Delta T_c} + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} \left(G_{ij} - H_i \frac{\pi a_j^2}{A_a}\right) \frac{q_j}{\Delta T_c} = 1$$
$$(i = 1\text{–}N) \tag{18}$$

Solving this system, we determine $(q_i/\Delta T_c)$ for $(i = 1, \ldots, N)$, and using Eqs. (12) and (15) we deduce the values of $R_{cs}^{(i)}$ and R_{cs} respectively.

3. Results

The present model is used to calculate the thermal constriction resistance due to a multiple contacts which radii are different and positions are randomly distributed over a square surface (the results of circular region are the same). The data (x_i, y_i,a_i) characterizing the contacts are generated by the random data of Matlab software. If a generated contact overlaps with an existing contact, it is omitted and replaced by the next one. We continue this procedure until the desirable number of contacts, N, is obtained. For each studied case we fix the relative contact size, $\varepsilon = \sqrt{A_r/A_a}$, the number of contacts, N, and the ratio of the largest disk radius, $\mathrm{Max}(a_i)$, to the smallest one, $\mathrm{Min}(a_i)$. This ratio is presented by the parameter RR, where $\mathrm{RR} = \mathrm{Max}(a_i)/\mathrm{Min}(a_i)$.

In a first step, we checked that the present model provides the same results as the existing model adapted to some identical and regular contacts.

In order to compare the random and regular contacts, we consider the number of contact, N, equal to a square number. The relative difference, E, between the two models is defined as follow:

$$E = \frac{R_{cs} - R_{cs}^{(u)}}{R_{cs}^{(u)}} \times 100\% \tag{19}$$

where $R_{cs}^{(u)}$ is the thermal constriction resistance of identical and regular contacts.

We have studied 36 cases combining three values of N (16, 49 and 100), four values of ε (0.1, 0.2, 0.3 and 0.4) and three values of RR (1, 5 and 10). The particular case $\mathrm{RR} = 1$ corresponds to identical disks configuration but randomly distributed. This allows to analyze the effect of contact position only. For each case, we plot the average value of 50 random sets realized in the same conditions.

The analytical solution is programed under Matlab software in order to simultaneously draw the disks of contacts and the individual thermal constriction resistance, $R_{cs}^{(i)}$, and calculate R_{cs} and E. Fig. 6 shows an example of presentation ($N = 49$, $\varepsilon = 0.218$ and $RR = 5$). In this example the relative difference between random and regular contacts, E, is equal to 29%. In Fig. 6(a) are plotted the results of random contacts and in Fig. 6(b) those of the equivalent identical and regular contacts. The results of random contact show an important dispersion of individual $R_{cs}^{(i)}$. The $R_{cs}^{(i)}$ decreases with the increasing of the disk radius, a_i, but some contacts have a larger radius and a higher constriction. This phenomenon is due to the interaction between contacts and the edge effect which are more or less important. The $R_{cs}^{(i)}$ values calculated by the present model for identical and regular contacts are the same and correspond exactly to those in the literature.

Fig. 7(a)–(c) shows the results for $\mathrm{RR} = 1$, 5 and 10 respectively. The R_{cs} evolution is plotted as a function of

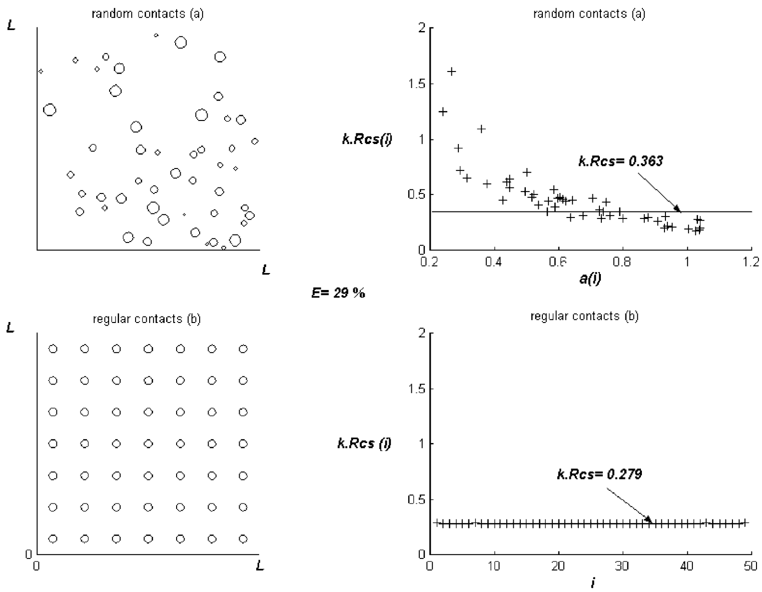

Fig. 6. Example of presentation of results under Matlab software.

ε for the three values of N. It is clearly shown that for all the studied cases the R_{cs} value of random contacts is greater than that for regular ones. For each value of the parameter RR the relative difference, E, increases with the increasing ε value. This evolution is almost linear. For equal value of ε, the relative difference, E, increases with the increasing of RR value. This means that the presence of asperities with very different sizes involve an increasing of thermal contact resistance value (the number of contact and the relative contact size being the same). Considering equal values of ε and RR, the number of contact N seems have a low influence on the thermal constriction resistance value.

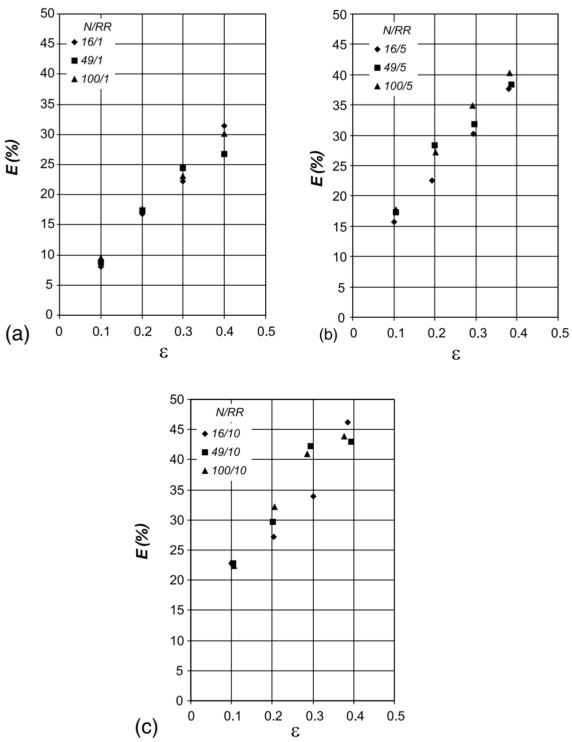

Fig. 7. Relative difference of constriction between random and regular contacts: (a) RR = 1, (b) RR = 5, (c) RR = 10.

The results of the three cases (RR = 1, 5 and 10) are shown in Fig. 8. The linear regressions corresponding to

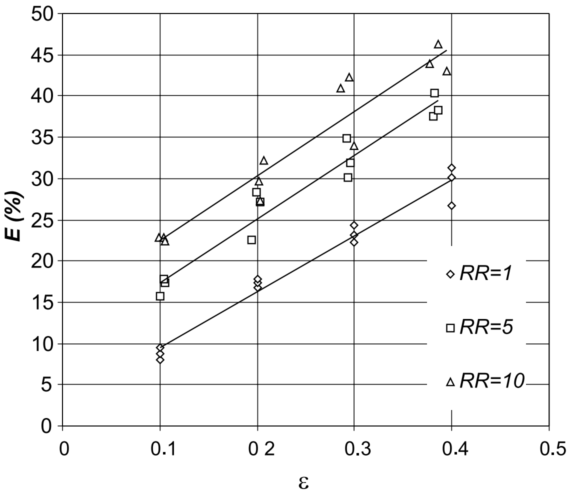

Fig. 8. Relative contact size and dispersion of disk radii effects on R_{cs}.

each case are slightly parallel. For RR values greater than 10 we almost obtained the same results as for case $RR = 10$.

4. Conclusions

An analytical solution is developed in this paper in order to calculate the thermal constriction resistance due to multiple disk contacts with random positions and sizes. This solution is easy to use and allows to consider numerous contacts. It provides accurate results with reasonable computational time. It is shown that, for equal relative contact size ε and number of contacts N, the thermal constriction resistance due to random contacts is systematically greater than the regular contacts. The evolution of relative difference E between the two configurations is almost linear as a function of ε. The value of E increases with the increasing of dispersion of disk radii (parameter RR) and stabilizes around the case $RR = 10$. For all cases the value of E are of the order of a few tens per cent.

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